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Autonomous ①

B.Tech - I sem. Regular Examinations, Dec/Jan-2024

Calculus and Differential equations

Solutions.

1a. $u = (x^2+y^2+z^2)^{-1/2}$

$$u_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = \frac{-x}{(x^2+y^2+z^2)^{3/2}} \longrightarrow (2m)$$

$$u_{xx} = -\left\{\frac{(x^2+y^2+z^2)^{3/2} - x \times \frac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}\right\}$$

$$= -(x^2+y^2+z^2)^{1/2}\left\{\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^3}\right\}$$

$$= -\frac{(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{5/2}} \longrightarrow (2m)$$

''ly $$u_{yy} = -\frac{(z^2+x^2-2y^2)}{(x^2+y^2+z^2)^{5/2}}$$

$$u_{zz} = -\frac{(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{5/2}} \quad \longrightarrow (2m)$$

$$\therefore u_{xx}+u_{yy}+u_{zz} = -\frac{1}{(x^2+y^2+z^2)^{5/2}}(y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2)$$

$$= 0. \longrightarrow (1m).$$

1.b. $u = x^2-2y,\quad v = x+y+z,\quad w = x-2y+3z$

$u_x = 2x,$	$v_x = 1$	$w_x = 1$
$u_y = -2$	$v_y = 1$	$w_y = -2$
$u_z = 0$	$v_z = 1$	$w_z = 3.$

$\longrightarrow (3m)$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \longrightarrow (2m)$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \longrightarrow (1m)$$

$$= 2x[3+2] + 2[3-1] - 0[-2-1]$$

$$= 10x + 4 \quad — (1m)$$

2(a). $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \quad — (1m)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \qquad \frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-(x-y)^2}} \longrightarrow (2m)$$

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 12t \quad — (1m)$$

$$\therefore \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} \times 3 - \frac{1}{\sqrt{1-(x-y)^2}} \times 12t \longrightarrow (2m)$$

$$= \frac{1}{\sqrt{1-(3t-4t^2)^2}} [3-12t^2] = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

$\hookrightarrow (1m)$

Q.b. $f(x,y) = x^2+3y^2-9x-9y+26$. about $(2,2)$ (2)

$f(x,y) = x^2+3y^2-9x-9y+26$ $\quad f(2,2) = 6.$

$f_x = 2x-9$ $\quad f_x(2,2) = -5$

$f_y = 6y-9$ $\quad f_y(2,2) = 3.$

$f_{xx} = 2$ $\quad$ (3m)

$f_{xy} = 0$

$f_{yy} = 6.$

The Taylor's series expansion of $f(x,y)$ about the point (a,b) is.

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

— (2m)

$$\therefore f(x,y) = 6 + (x-2)(-5) + (y-2)(3) + \frac{1}{2!}\left[2(x-2)^2 + 2(x-2)(y-2)\times 0 + (y-2)^2\times 6\right]$$

$$\Rightarrow f(x,y) = 6-5(x-2)+3(y-2)+(x-2)^2+3(y-2)^2.$$

— (2m)

3a. $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

$f_x = 4x^3-4x+4y$, $f_y = 4y^3+4x-4y$. $\rightarrow$ $\boxed{1m}$

$f_x = 0$, $f_y = 0$ $\Rightarrow$ $4x^3-4x+4y = 0 \rightarrow$ (1)

$4y^3+4x-4y = 0 \rightarrow$ (2)

(1) + (2) $\Rightarrow$ $x^3+y^3=0 \Rightarrow \boxed{x=-y.}$ $\longrightarrow$ $\boxed{1m}$

subl. in eqn (1) we get $4x^3-4x-4x = 0$

$\Rightarrow 4(x^3-2x) = 0$

$\Rightarrow 4x(x^2-2) = 0$

$\Rightarrow x = 0, x = \pm\sqrt{2}$

The corresponding values of y are $0, -\sqrt{2}, \sqrt{2}$

$\therefore$ The stationary points are $(0,0)$ $(\sqrt{2},-\sqrt{2})$ $(-\sqrt{2},\sqrt{2})$. $\longrightarrow$ $\boxed{1m}$

$r = f_{xx} = 12x^2-4$, $s = f_{xy} = 4$, $t = f_{yy} = 12y^2-4$ $\longrightarrow$ $\boxed{1m}$

At $(0,0)$ $r=-4$, $s=4$, $t=-4$

$rt-s^2 = 16-16 = 0$ it need further investigation

$\longrightarrow$ $\boxed{1M}$

At $(\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$, $r = 20$, $s=4$, $t=20$.

$rt-s^2 = 400-16 > 0$ & $r = 20 > 0$

f attains its minimum at $(\sqrt{2},-\sqrt{2})$ & $(-\sqrt{2},\sqrt{2})$ & the minimum value is $f(+\sqrt{2},-\sqrt{2}) = \cancel{4}+\cancel{4}-\cancel{4}-8-4 = -8$

$\longrightarrow$ $\boxed{2m}$

3b. $f = x^2+y^2+z^2, \quad xyz = 216 \rightarrow ①$

let $\phi = xyz - 216 = 0$

The Lagrangean function is $F = f + \lambda\phi \rightarrow$ (1m)

i.e. $F = x^2+y^2+z^2+\lambda(xyz-216)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 0 \Rightarrow 2x+\lambda yz = 0 \Rightarrow \lambda = \frac{-2x}{yz} \\ \frac{\partial F}{\partial y} = 0 \Rightarrow 2y+\lambda xz = 0 \Rightarrow \lambda = \frac{-2y}{xz} \\ \frac{\partial F}{\partial z} = 0 \Rightarrow 2z+\lambda yx = 0 \Rightarrow \lambda = \frac{-2z}{xy} \end{array}\right\} \text{(3m)}$$

$\therefore \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = -\frac{\lambda}{2} \longrightarrow$ (1m)

$\Rightarrow x^2 = y^2 = z^2 \Rightarrow x = y = z \longrightarrow$ (1m)

Subs. in ① we get

$x^3 = 216 \Rightarrow x = 6$

$\therefore x = y = z = 6 \longrightarrow$ (1m)

Hence the function attains its Extremum

at $(6,6,6)$

4.a $f = x^3+y^3-3axy$.

$f_x = 3x^2 - 3ay \; ; \; f_y = 3y^2 - 3ax \longrightarrow \boxed{1m}$

$f_x = 0, f_y = 0 \Rightarrow 3(x^2-ay) = 0 \rightarrow ①$

$3(y^2-ax) = 0 \rightarrow ②$

$x^2 - ay = 0 \rightarrow (1) \quad y^2 - ax = 0 \rightarrow (2)$

$(1) - (2) \Rightarrow \quad x^2 - y^2 + a(x - y) = 0$

$\Rightarrow (x-y)(x+y+a) = 0$

$\Rightarrow \boxed{x = y} \longrightarrow (1m)$

Subt. in eq (1) we get $x(x-a) = 0$

$\Rightarrow x = 0, x = a.$

when $x = 0, x = a$, $y = 0, y = a$ respectively

$\therefore$ The Stationary points are $(0,0)$ (a,a). $\longrightarrow \boxed{1M}$

Now $r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$.

At $(0,0)$, $r = 0$, $s = -3a$, $t = 0$ $\longrightarrow \boxed{1m}$

$\therefore rt - s^2 = -9a^2 < 0$.

$(0,0)$ is a saddle point, f attains neither its maximum nor its minimum at $(0,0)$

At (a,a), $r = 6a$, $s = -3a$, $t = 6a$ $\longrightarrow \boxed{1m}$

$\therefore rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$ &

If $a > 0$, $r = 6a > 0$ then f attains its minimum (a,a). $\longrightarrow \boxed{1m}$

If $a < 0$ then $r = 6a < 0$ then f attains its maximum at (a,a). $\longrightarrow \boxed{1m}$

4.b Let $f = x^2+y^2+z^2$, $ax+by+cz = p \rightarrow (1)$

$$\phi = ax+by+cz-p = 0.$$

The Lagrangean function is $F = f+\lambda\phi \rightarrow \boxed{1M}$

$$\therefore F = x^2+y^2+z^2+\lambda(ax+by+cz-p)$$

$$\left.\begin{aligned}
\therefore \frac{\partial F}{\partial x} = 0 &\Rightarrow 2x+a\lambda = 0 \Rightarrow \lambda = -\frac{2x}{a} \\
\frac{\partial F}{\partial y} = 0 &\Rightarrow 2y+b\lambda = 0 \Rightarrow \lambda = -\frac{2y}{b} \\
\frac{\partial F}{\partial z} = 0 &\Rightarrow 2z+c\lambda = 0 \Rightarrow \lambda = -\frac{2z}{c}
\end{aligned}\right\} \rightarrow \boxed{3M}$$

$$\therefore \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{-\lambda}{2} \rightarrow \boxed{1M}$$

Subl. in eq(1) we get

$$ax + \frac{b^2x}{a} + \frac{c^2x}{a} = p$$

$$\Rightarrow (a^2+b^2+c^2)x = ap.$$

$$\left.\begin{aligned}
\Rightarrow x &= \frac{ap}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2} \\
y &= \frac{bp}{a^2+b^2+c^2}
\end{aligned}\right\} \rightarrow \boxed{1M}$$

$\therefore$ At $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$ function attains its minimum & the minimum value is $\frac{p^2}{a^2+b^2+c^2}$ — $\boxed{1M}$

5a. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\, dx \longrightarrow \boxed{1M}$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy.(1-x^2-y^2)\, dy\, dx \longrightarrow \boxed{1M}$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left(x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right)_0^{\sqrt{1-x^2}} dx \longrightarrow \boxed{1M}$$

$$= \frac{1}{2}\int_0^1 \left[\left(\frac{x}{2} - \frac{x^3}{2}\right)(1-x^2) - \frac{x}{4}(1-x^2)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x}{2}(1-x^2)^2 - \frac{x}{4}(1-x^2)^2\right) dx \longrightarrow \boxed{2M}$$

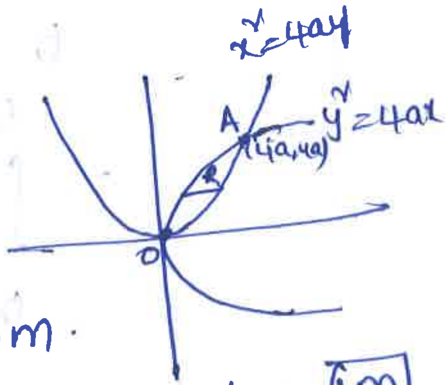
$$= \frac{1}{8}\int_0^1 x(1-x^2)^2\, dx = \frac{1}{8}\int_0^1 x[1+x^4-2x^2]\, dx$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\,dx$$

$$= \frac{1}{8}\left\{\frac{x^2}{2} - 2\frac{x^4}{4} + \frac{x^6}{6}\right\}_0^1 = \frac{1}{8}\left\{\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right\} = \frac{1}{48}$$

$\rightarrow \boxed{2M}$

5.b. Given $y = \frac{x^2}{4a}$, $y = 2\sqrt{ax}$



$\Rightarrow x^2 = 4ay$, $y^2 = 4ax$.

$\rightarrow \boxed{1M}$

over the region R : x varies from

$\frac{y^2}{4a}$ to $2\sqrt{ay}$. $\rightarrow \boxed{2M}$

y varies from 0 to 4a.

$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}\,dy$$

$\rightarrow \boxed{2M}$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= 2\sqrt{a}\left\{\frac{y^{3/2}}{3/2}\right\}_0^{4a} - \frac{1}{4a}\left\{\frac{y^3}{3}\right\}_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left\{(4a)^{3/2}\right\} - \frac{1}{12a}\left\{4^3a^3\right\}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3} \longrightarrow \boxed{2m}$$

6.a. $$\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy \longrightarrow \boxed{1m}$$

$$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\,dy\,dx$$

$$= \int_0^5 \left(x^3 y + x\frac{y^3}{3}\right)_0^{x^2} dx = \int_0^5 \left(x^5 + \frac{x}{3}x^6\right) dx \longrightarrow \boxed{3m}$$

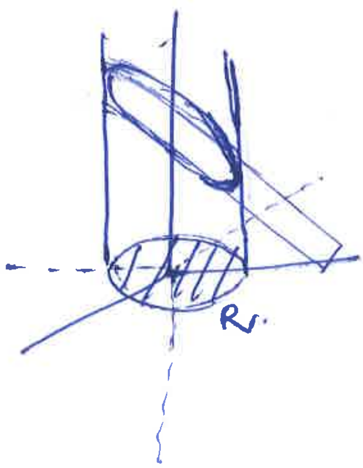
$$= \left(\frac{x^6}{6} + \frac{1}{3}\frac{x^8}{8}\right)_0^5 = \frac{1}{6}[5^6] + \frac{1}{24}[5^8]$$

$$= 5^6\left[\frac{1}{6} + \frac{25}{24}\right]$$

$$= 5^6\left[\frac{29}{24}\right] \longrightarrow \boxed{3m}$$

6.b. The required volume is $= \iint_R z\,dx\,dy \longrightarrow \boxed{1m}$

$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy \longrightarrow \boxed{2m}$$



$$= 2\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} (4-y)\,dx\,dy \longrightarrow \boxed{1m}$$

$$= 2\int_{-2}^{2} (4-y)\cdot(x)_0^{\sqrt{4-y^2}}\,dy$$

$$= 2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy \longrightarrow \boxed{1M}$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$= 8\times 2\int_{0}^{2}\sqrt{4-y^2}\,dy - 0 \longrightarrow \boxed{1M}$$

$$= 16\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2$$

$$= 16\left[0 + 2\times\frac{\pi}{2}\right] = 16\pi \longrightarrow \boxed{1M}$$

7.a. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0 \rightarrow (1)$

Eq (1) is in the form $Mdx + Ndy = 0$

$M = x^2y - 2xy^2$, $N = 3x^2y - x^3$

$\frac{\partial M}{\partial y} = x^2 - 4xy$, $\frac{\partial N}{\partial x} = 6xy - 3x^2 \longrightarrow \boxed{1M}$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so Eq (1) is not Exact $\longrightarrow \boxed{1M}$

clearly M & N are homogeneous functions of same degree.

& $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

$\therefore$ I.F $= \frac{1}{Mx+Ny} = \frac{1}{x^2y^2} \longrightarrow \boxed{1M}$

multiply Eq(1) by I.F $= \frac{1}{x^2y^2}$ we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0 \rightarrow (2)$$

$\rightarrow \boxed{1M}$

clearly Eq(2) is Exact $\left[\frac{\partial m_1}{\partial y} = -\frac{1}{y^2}\right.$

$\therefore$ G.S is $\int\limits_{y\,const} M_1\,dx + \int \left(\text{terms of } N_1 \text{ not containing } x\right) dy$ $\left.\frac{\partial n_1}{\partial x} = -\frac{1}{y^2}\right]$

$= C. \rightarrow \boxed{1M}$

$$\int\limits_{y\,const} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = C$$

$\Rightarrow \frac{x}{y} - 2\ln x + 3\ln y = C$ ie., $\frac{x}{y} + \ln\left(\frac{y^3}{x^2}\right) = C.$

$\rightarrow \boxed{2m}$

=

7.b $\quad (D^2 - 2D + 1)y = \frac{e^x}{x}.$

$f(D)y = R.$

$\therefore$ The A.E is $f(m) = 0 \Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore y_c = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x \rightarrow \boxed{1m}$

$= c_1 u + c_2 v.$

where $u = e^x$, $v = xe^x \rightarrow (1M)$

(7)

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$$

$$= e^{2x} \neq 0$$

$\longrightarrow \boxed{1M}$

Take $y_p = Au + Bv = Ae^x + Bxe^x$

where $A = -\int \frac{vR}{W} dx$, $B = \int \frac{uR}{W} dx$ — $\boxed{1M}$

$$A = -\int \frac{xe^x \frac{xe^x}{x}}{e^{2x}} dx = -\int 1\,dx = -x$$

$$B = \int \frac{e^x \frac{xe^x}{x}}{e^{2x}} dx = \int \frac{1}{x} dx = \ln x.$$

$\rightarrow \boxed{2M}$

$\therefore y_p = -xe^x + (\ln x) x e^x.$

$\therefore$ The G.S is $y = y_c + y_p = (c_1 + c_2 x)e^x - xe^x + xe^x \ln x$

— $\boxed{1M}$

8a. $(D^2 - 1)y = e^x + x^2 e^x.$

$f(D)y = Q$ where $f(D) = D^2 - 1$, $Q = e^x + x^2 e^x$

A.E is $f(m) = 0$ ie., $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore y_c = c_1 e^x + c_2 e^{-x}$ where c_1, c_2 are arbitrary

$\longrightarrow \boxed{2M}$

$$y_p = \frac{1}{f(D)} Q = \frac{1}{D^2 - 1} e^x + \frac{1}{D^2 - 1} e^x x^2$$

$= I_1 + I_2$

where $I_1 = \frac{1}{D^2-1} e^x$

$= \frac{x}{2D} e^x = \frac{xe^x}{2}$. $\rightarrow \boxed{2m}$

$I_2 = \frac{1}{D^2-1} e^x x^2$

$= e^x \frac{1}{(D+1)^2-1} x^2$

$= e^x \frac{1}{D^2+2D} x^2$.

$= e^x \frac{1}{2D\left(1+\frac{D}{2}\right)} x^2$

$= \frac{e^x}{2} \cdot \frac{1}{D}\left(1+\frac{D}{2}\right)^{-1} x^2$

$= \frac{e^x}{2} \frac{1}{D}\left[1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \cdots\right] x^2$

$= \frac{e^x}{2}\left\{\frac{1}{D} - \frac{1}{2} + \frac{D}{4} - \frac{D^2}{8} + \cdots\right\} x^2$.

$= \frac{e^x}{2}\left\{\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4} \times 2x - \frac{1}{8} \times 2\right\}$.

$= \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2} - \frac{1}{4}\right] \longrightarrow \boxed{2m}$

$\therefore y_p = \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2} - \frac{1}{4}\right)$

Hence the G.S is $y = y_c + y_p$.

$\therefore y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2} - \frac{1}{4}\right]$

$\longrightarrow \boxed{1m}$

8.b. By Newton's Law of cooling the rate of change in temp. of a body is directly proportional to the diff. in temperature of the body and that of surrounding medium. ⟶ $\boxed{1m}$ (8)

Let 'θ' be the temp. of a body at time t.

& θ_0 be the temp. of the surrounding medium.

$$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -K(\theta - \theta_0) \longrightarrow (1)$$

By solving (1) we get $\underline{\theta = \theta_0 + Ce^{-kt}}$. ⟶ $\boxed{1m}$

Given $\theta_0 = 40^\circ C$, $\theta = 80^\circ C$ when $t = 0$

$$\therefore 80 = 40 + Ce^{-K\times 0} \Rightarrow \boxed{C = 40} \text{ — } \boxed{1m}$$

when $\theta = 60^\circ$, $t = 20$ min.

$$\therefore 60 = 40 + 40e^{-K\times 20}$$

$$\Rightarrow \frac{1}{2} = e^{-20K} \Rightarrow \boxed{K = -\frac{1}{20}\ln\left(\frac{1}{2}\right)}$$

$$\therefore \theta = 40 + 40e^{\frac{1}{20}\ln\left(\frac{1}{2}\right)t} \longrightarrow \boxed{2m}$$

When $t = 40$ min.

$$\theta = 40 + 40e^{\frac{1}{20}\ln\left(\frac{1}{2}\right)\times 40}$$

$$= 40 + 40e^{2\ln\left(\frac{1}{2}\right)}$$

$$= 40 + 40\times\left(\frac{1}{2}\right)^2 = \underline{\underline{50^\circ C}} \longrightarrow \boxed{2m}$$

9.a. Let $f(t) = \cos at - \cos bt$

$$L[f(t)] = L[\cos at] - L[\cos bt]$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = F(s) \rightarrow \boxed{2m}$$

WKT $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \rightarrow \boxed{1m}$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\Bigg]_s^\infty \rightarrow \boxed{2m}$$

$$= \frac{1}{2}\left\{0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\} \rightarrow \boxed{1m}$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \rightarrow \boxed{1m}$$

9.b. Given eqn. is $y'' + 4y' + 3y = e^{-t}$

$$L[y''] + 4L[y'] + 3L[y] = L[e^{-t}]$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4[s\,Y(s) - y(0)] + 3Y(s)$$

$$= \frac{1}{s+1}$$

$$(s^2+4s+3)Y(s) - s - 1 - 4 = \frac{1}{s+1}$$

$$(s^2+4s+3)Y(s) = \frac{1}{s+1} + s+5$$

$$\therefore Y(s) = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{s^2+4s+3}.$$

$$= \frac{1}{(s+1)^2(s+3)} + \frac{s+5}{(s+1)(s+3)}.$$

$$= \frac{1+(s+5)(s+1)}{(s+1)^2(s+3)}$$

$$Y(s) = \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$\therefore s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

put $s=-1$

$$\therefore 1 = B(2) \Rightarrow \boxed{B = 1/2}$$

$9-18+6$

put $s=-3$.

$$-3 = C(4) \Rightarrow \boxed{C = -3/4}$$

comparing the coeff. of like powers on bs weget

$$A + C = 1 \Rightarrow A = 1 - C = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\therefore Y(s) = \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2} - \frac{3}{4(s+3)}.$$

Apply ILT on bs we get

$$\therefore L^{-1}[Y(s)] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$\therefore y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}t - \frac{3}{4}e^{-3t}.$$

10.a. let $f(t) = \sin 3t$

$$\therefore L[f(t)] = L[\sin 3t] = \frac{3}{s^2+9} = F(s) \quad \text{— } \boxed{1M}$$

WKT $L[t f(t)] = -\frac{d}{ds}[F(s)]$ — $\boxed{1M}$

$$\therefore L[t \sin 3t] = -3 \times \frac{-2s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2} \quad \text{— } \boxed{1M}$$

By def^n. $\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$ — $\boxed{1M}$

$$\therefore \int_0^\infty e^{-2t}\, t \sin 3t\, dt = L[t \sin 3t]$$

where $s = 2$ — $\boxed{1M}$

$$= \left[\frac{6s}{(s^2+9)^2}\right]_{s=2}$$

$$= \frac{6\times 2}{(4+9)^2} = \frac{12}{169} =$$

— $\boxed{2M}$

10.b. let $F(s) = \frac{1}{s^2+1}$, $G(s) = \frac{1}{s^2+9}$

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$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = f(t)$$
$$L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin 3t = g(t)$$ — $\boxed{2m}$

By convolution theorem. $L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)\,du \rightarrow \boxed{1m}$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = \int_0^t \sin u \times \frac{1}{3}\sin 3(t-u)\,du \rightarrow \boxed{1m}$$

$$= \frac{1}{6}\int_0^t \cos[u-3t+3u] - \cos[u+3t-3u]\,du \rightarrow \boxed{1m}$$

$$= \frac{1}{6}\left[\int_0^t [\cos(4u-3t) - \cos(3t-2u)]\,du\right]$$

$$= \frac{1}{6}\left\{\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right\}_0^t \rightarrow \boxed{1m}$$

$$= \frac{1}{6}\left\{\frac{1}{4}[\sin t + \sin 3t] + \frac{1}{2}[\sin t - \sin 3t]\right\}$$

$$= \frac{1}{6}\left\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right\}$$ — $\boxed{1m}$.

$$= \frac{1}{24}[3\sin t - \sin 3t].$$

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$\frac{1}{8}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right)$